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RESEARCH REPORT No. EM-83

Equivalence Relations in Diffraction Theory

S. N. KARP and W. ELWYN WILLIAMS

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Morris Kline
Morris Kline
Project Director

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Abstract

Certain equivalence relationships between different problems occurring in diffraction theory are established. The method used is essentially Schwartz principle of analytic continuation by reflection across a straight boundary. An explicit solution is also obtained by this method for a plane wave normally incident on a T-shaped structure.

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1. Introduction

It is of some interest in mathematical physics to find methods whereby the solution of one boundary-value problem may be deduced from the solution of a different one, i.e., to establish equivalence relationships between different problems.

One example of the utility of equivalence relations between different problems is afforded by Kelvin's well-known method of images where the original boundary-value problem is transformed into one for a larger domain for which a solution is generally available by inspection. Babinet's principle [1] in diffraction theory is a further example of the utility of equivalence relationships between different boundary-value problems. It is of interest to note that the process used in the present paper may be applied to deduce Babinet's principle.

The object of the present paper is to obtain some more general equivalence relationships between different problems occurring in acoustic and electromagnetic diffraction theory. The first problem considered is that of the diffraction of a plane wave normally incident on a two-dimensional infinite grating composed of strip elements equal in width to the grating spacing. This problem was recently solved by Baldwin and Heins [2], who, from symmetry considerations, reduced it to the diffraction problem for an iris in a parallel-plate waveguide. They then obtained a solution to this latter problem by means of the Wiener-Hopf technique, which has not previously proved useful for the wave diffraction problems involving finite obstacles. It is shown in the present paper, however, that this solution may be deduced immediately from the solution of the diffraction problem for a bifurcated waveguide (or, equivalently, from the diffraction problem for an infinite set of semi-infinite parallel plates, which has been solved previously by Carlson and Heins [3]). The equivalence relationship between these two apparently different problems is then generalized to cover more complicated geometries, both for closed domains and for free space. In particular, it is shown that the solution for the diffraction of a plane wave by a T-shaped obstacle, when the vertical part is infinite, may be expressed in terms of the solutions of the two independent diffraction problems: the problem of a semi-infinite half-plane and the problem of a strip equal in width to the horizontal part of the original obstacle. For the special case of incidence normal to the horizontal part* it is possible to write down an explicit solution to the problem in terms of Fresnel integrals. Finally, we establish the equality of the eigenvalues of two apparently different eigenvalue problems.

All the problems considered here have solutions which are defined in in some domain and are analytic everywhere in that domain (which may be unbounded) except across a certain plane surface R, say, where either the function or its normal derivative vanishes. The procedure used here to establish the equivalence relationships described above is to take such a function and continue it analytically across R by even or odd reflection (as the case may be) up to some new surface R'. The new function so derived will not be analytic across R' and will not, in general, satisfy any simple boundary condition on R'. However, in the problems considered here it is possible by invoking certain properties of symmetry, to deduce from this function a solution satisfying both the differential equation and some simple boundary condition on R'; hence this is the solution to a new boundary-value problem and the equivalence of the two problems is thus established.

Thus the process used here is essentially the application of Schwartz' reflection principle to the continuation of the solution into a second sheet of its Riemann surface.

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* In fact for any incident field symmetric with respect to the vertical part of the obstacle.

2. Diffraction by a grating

In this section we consider the problem of the diffraction of a plane wave by a grating described in the Introduction. Due to the symmetry of the initial configuration it is clearly equivalent to the diffraction problem in a parallel-plate waveguide when a thin obstacle is placed normal to the waveguide axis on one of the two plates and the height of the obstacle is half that of the waveguide (Fig. 1). The field incident on this configuration is the fundamental mode for the parallel-plate region.

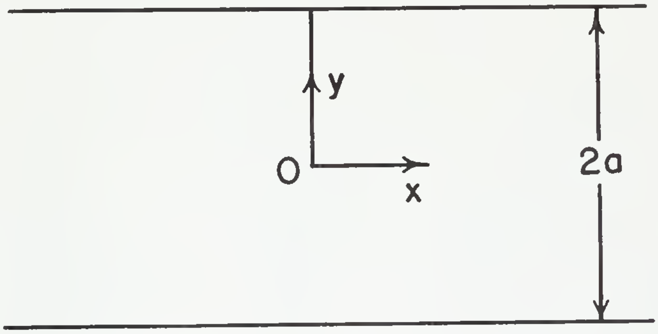

Figure 1

The problem considered is to determinae a solution of $(\nabla^2 + k^2)u = 0$, where $\nabla^2 = \partial^2/\partial^2 dx^2 + \partial^2/\partial^2 dy^2$, which has zero normal derivative for all x if $y = \pm a$ and for $y = 0$ if $0 < y < a$, and which is such that $u - e^{ikx}$ is an outgoing wave at infinity. A time dependence $e^{i\omega t}$ is suppressed. For convenience we number the regions of the (x,y)-plane as follows:

$$-a \leq y \leq 0, \quad x \geq 0 \quad = \quad \text{Region I} \;,$$

$$-a \leq y \leq 0, \quad x \leq 0 \quad = \quad \text{Region II} \;,$$

$$0 \leq y \leq a, \quad x \leq 0 \quad = \quad \text{Region III},$$

$$0 \leq y \leq a, \quad x \geq 0 \quad = \quad \text{Region IV} \;.$$

As a preliminary to the discussion we define an auxillary function $v(x,y)$ to be a solution of the wave equation which behaves like e^{ikx} as $x \to -\infty$ and has zero normal derivative on $y = \pm a$ and vanishes on $y = 0$ for $x > 0$. This is essentially the function obtained by Carlson and Heins and it is also given in the Wave Guide Handbook[4]. This function $v(x,y)$ is now continued analytically across $y = 0$, $x \geq 0$: we define a new function $v^*(x,y)$ to be $v(x,y)$ in Regions I to III and $-v(x,-y)$ in Region IV. This new function is analytic everywhere in Regions I to IV except on $x = 0$ for $0 \leq y \leq a$. By reflecting $v^*(x,y)$ oddly in the plane $x = 0$ and adding this new function to $v^*(x,y)$ we define a new function $w(x,y)$; the expressions for $w(x,y)$ in terms of $v(x,y)$ are:

$$w(x,y) = \begin{cases} v(x,y) - v(-x,y) & \text{for Regions I and II} \\ v(x,y) + v(-x,+y) & \text{for Regions III} \\ -v(x,-y) - v(-x,-y) & \text{for Regions IV} \;, \end{cases}$$

Clearly $w(x,y)$ is analytic everywhere except on $x = 0$, $0 \leq y \leq a$. By symmetry $v(x,y) = v(x,-y)$, and hence we see that $w(x,y)$ is discontinuous across the strip $x = 0$, $0 \leq y \leq a$ while $\partial w/\partial x$ vanishes on both sides of this strip. Thus the function $w(x,y)$ represents a solution of the wave equation which satisfies all the boundary conditions imposed on the original function u except those at infinity. Clearly, then, $w(x,y)$ represents the superposition of two solutions of the type $u(x,y)$ arising respectively from a field e^{ikx} incident from $x = -\infty$ and a field $-e^{-ikx}$ incident from $x = \infty$. However, we require to determine the solution due to only one incident wave.

It is clearly possible to write $u(x,y)$ as $e^{-ikx} + e^{+ikx} - \Psi(x,y)$ for $x \leq 0$, and as $\Psi(-x,y)$ for $x \geq 0$, where $\Psi(x,y)$ is to be determined. The boundary conditions to be satisfied by $\Psi(x,y)$ are obvious. We thus see immediately that $w(x,y)$ may be expressed in terms of $\Psi(x,y)$ and vice versa. For $x < 0$ we thus have $2\Psi(x,y) = e^{-ikx} + e^{ikx} - w(x,y)$. So the complete solution of the first problem has now been obtained explicitly in terms of $v(x,y)$. It is of interest to obtain the reflection coefficients for the function $u(x,y)$. For the fundamental mode one finds, in view of the above, that this is $(R+1)/2$, where R is the corresponding reflection coefficient for $v(x,y)$. In general the coefficient of the n-th mode of $u(x,y)$ is half of that for $v(x,y)$.

If the original function $v^*(x,y)$ has been reflected evenly in the plane $x = 0$, then a solution would have been obtained which vanishes on the strip $x = 0$, $0 \leq y \leq a$ and which has zero normal derivative on $y = \pm a$.

It should also be observed that a function $w'(x,y)$ which apparently satisfies all the same conditions as $w(x,y)$ would have been obtained by using, instead of $v(x,y)$, a function $v'(x,y)$ which has zero normal derivative on the region $y = 0$, $x \geq 0$. In this case the continuation of $v'(x,y)$ across the bifurcation would have been given by even reflection and it would then be necessary to reflect the resulting function evenly across the plane $x = 0$. However,

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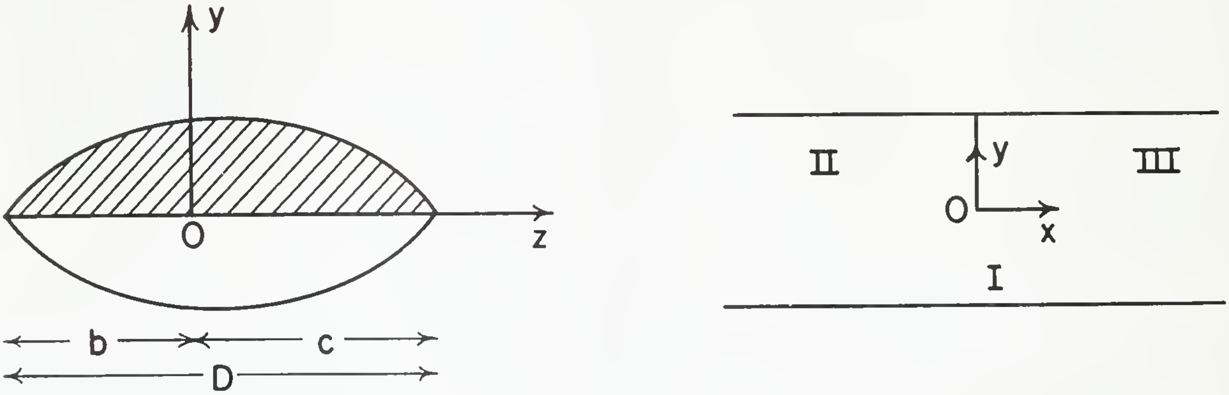
for an incident field e^{ikx}, the solution $v'(x,y)$ is trivial namely, e^{ikx} itself, and thus $w'(x,y)$ would equal $e^{ikx} + e^{-ikx}$. If, however, a solution were required for $u(x,y)$ when the incident part is odd about $y = 0$ then a solution of the type $v'(x,y)$ would have to be used.

3. Generalization of results

In this section the above results are generalized to cover more complicated geometries. The above analysis is clearly valid for geometries other than that of the parallel-plate type. Instead of considering the interior of a parallel-plate waveguide it is, for instance, permissible to consider the interior of a cylinder of uniform but arbitrary cross-section which is symmetric about a diameter D, the cylinder being assumed to extend from $x = -\infty$ to $x = +\infty$ (see Fig. 2).



cross-sectional view

side view

Figure 2

The origin of a Cartesian system of coordinates is taken on D, and that part of the cylinder cross-section defined by $x = 0$, $y \geq 0$ will be denoted by S. The dependence of all functions concerned on the third coordinate z is suppressed throughout.

The above analysis now shows that a solution of the wave equation satisfying a homogeneous type of boundary condition on the cylinder and on S may be obtained from a solution of the wave equation satisfying this same boundary condition on the cylinder but satisfying a homogeneous boundary condition on the strip $y = 0$, $x \geq 0$ instead of on S. The width of the strip is the diameter D (i.e., $-b \leq z \leq c$ on the strip) and the condition to be satisfied on it is to be such that the resulting solution is not the trivial one, i.e., the incident field itself. If these problems are denoted by A and B respectively then it is possible to represent the relationship between them symbolically as follows.

We define an superscript i attached to a function to mean that if $i = 1$ the solution vanishes on the cylinder while if $i = 2$ the normal derivative vanishes. The subscript j will indicate the same behavior of the function except that the boundary values are taken on S. For convenience of notation the cylinder region is divided into three regions as shown in Fig. 2; these are indicated by superscript $r = 1,2,3$. We define two matrives V and C by

$$V = \begin{pmatrix} v^{(1)}(x,y), & v^{(1)}(-x,y), \\ v^{(2)}(x,y), & v^{(2)}(-x,y) \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix},$$

where the $v^{(i)}$ are appropriate non-trivial solutions of problem B. The product VC is written as $B^{(1)}, B^{(2)}, B^{(3)}$, where the $B^{(r)}$ are 2×2 matrices.

Then a solution u_j^i of problem A is given in the r-th region by the ij-th element of $B^{(r)}$. The incident field producing u_j^i is again a superposition of two incident fields in exactly the same manner as the solution obtained in Section 2. A solution involving only one incident field may be isolated by the same method as the one used there.

It should be noted that in the above work it is not necessary, in continuing a solution across the semi-infinite strip, to retain the origin of the x-coordinates on the edge of the strip. It is clear that carrying out this process with the origin displaced a distance a to the right of the strip edge would yield a function analytic everywhere except on S and on $-a \leq x \leq a$, $y = 0$, $-b \leq z \leq c$. The resulting function would also satisfy a homogeneous boundary condition on this surface. From a solution of the boundary-value problem for the semi-infinite strip in the cylinder region it is thus possible to obtain a) a solution which vanishes on both S and on $y = 0$, for $-a \leq x \leq a$, $-b \leq z \leq c$; b) a solution which has zero normal derivative on both these boundaries; or c) a solution which vanishes on one boundary and has zero normal derivative on the other.

Unfortunately, however, such solutions a), b), c) involve two incident waves coming in from $x = \pm\infty$ respectively, and it does not appear to be possible to obtain a solution for the case when only one wave is incident on the T-shaped structure. The above approach, however, can be utilized to show that the solution to the problem of one incident wave may be deduced from the solutions of the two separate boundary-value problems for the finite strip $y = 0$, $-a \leq x \leq a$, $-b \leq z \leq c$ (which we shall denote by S') and the semi-finite strip $y = 0$, $x \geq 0$, $-b \leq z \leq c$ respectively. By the previous analysis the solution for the second of these two boundary-value problems is equivalent to that for S, that is, for the surface $x = 0$, $0 \leq y \leq a$, $-b \leq z \leq c$. Thus we see that the desired solution to the diffraction problem for the T-shaped structure may be represented in terms of the diffraction problem for its constituent parts S and S'. The relationship between these three problems will now be shown. For simplicity we consider the case where the function is to vanish on T.

Let $u^*(x,y,z)$ be the solution for the problem of a wave $u_0(x,y,z)$ incident from the left on the horizontal S' strip. Then it is clear that the desired solution $u(x,y,z)$ vanishing everywhere on the T may be expressed in terms of one unknown function $\Psi(x,y,z)$ by writing

$$\begin{aligned} u(x,y,z) &= u^*(x,y,z) - u^*(-x,y,z) + \Psi(x,y,z), & x \leq 0 \\ &= \Psi(-x,y,z), & x \geq 0 . \end{aligned}$$

On the other hand, as noted earlier, the solution a), which was obtained from the solution for the semi-infinite strip, involves two incident waves of equal strength incident from $x = \infty$ and $x = -\infty$ respectively. Hence this solution is given by $u^*(x,y,z) - u^*(-x,y,z) + 2\Psi(x,y,z)$ or

$$2u(x,y,z) - \left\{ u^*(x,y,z) - u^*(-x,y,z) \right\} .$$

The equivalence of the three problems is thus established.

The application of the above procedure is not restricted to bounded regions and it may be applied to free-space boundary-value problems. If we take the solution of the wave equation which vanishes on the half-plane $x \geq 0$, $y = 0$ (the incident field being symmetric with respect to the plane) in the region $y \leq 0$ and then continue this analytically across the half-plane by odd reflection and then reflect this solution evenly about $x = 0$, then a solution is obtained which vanishes on a half-plane $x = 0$, $y \geq 0$. This is of course a trivial result, but by choosing a new x-origin at distance a to the right of the old one and continuing analytically the solution of the original half-plane problem across $x \geq 0$, $y = 0$ and then proceeding as above, we obtain a more general result. This process is seen to define a function vanishing on $x = 0$ for $y \geq 0$ and on $y = 0$ for $-a \leq x \leq a$. The field incident on this new structure will be symmetric with respect to $y = 0$, so that in general our procedure does not result in solutions for one field incident on this structure. As before, however, it is possible to show that such a solution may be obtained in terms of the solutions for the diffraction problems for the strip $-a \leq x \leq a$, $y = 0$ and the half-plane $x \geq 0$, $y = 0$. Thus we see that for this structure too the solution to the diffraction problem for one incident wave may be expressed in terms of its two constituent parts; in a sense this constitutes an isolation of the effect of 'thickness' (see discussion below).

There is one special incident field for which an explicit solution may be given for the T-shaped structure. This is the solution obtained when the half-plane solution that we start with is the one for two plane waves of equal strength incident from $y = \pm\infty$ respectively. If the solution of this problem is denoted by $u(x,y)$ then the solution $v(x,y)$ for the diffraction of a wave normally incident on the cross-bar of the T from $y = -\infty$ is given by

$$\begin{aligned} v(x,y) &= u(x,y) + u(-x,y), && y \leq 0 , \\ &= u(x,y) - u(-x,-y), && x \leq 0 , \quad y \geq 0 , \\ &= u(-x,y) - u(x,-y) , && x \geq 0, \quad y \geq 0 . \end{aligned}$$

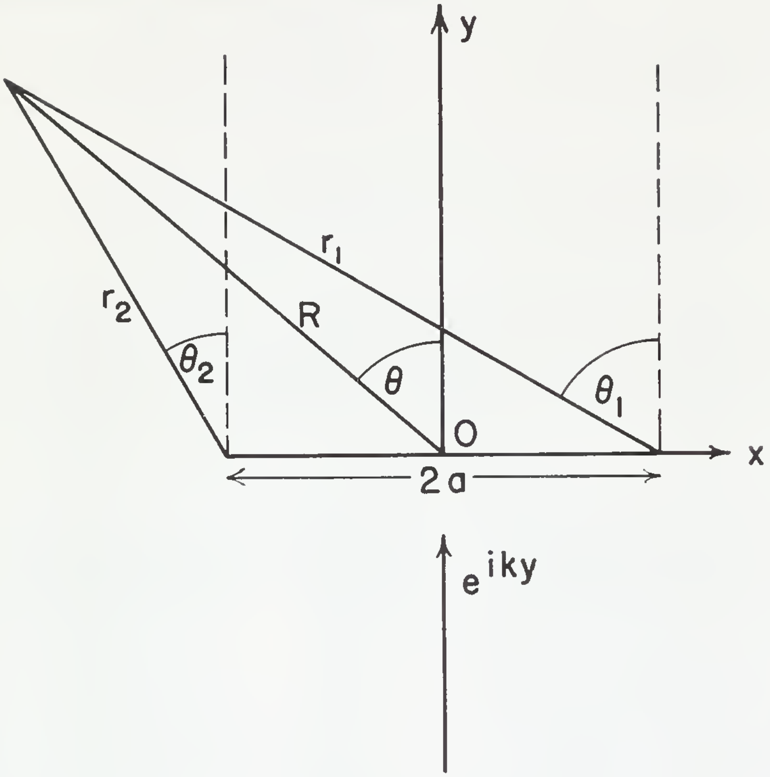


Figure 3

The solution $u(x,y)$ of our half-plane problem may be written explicitly in terms of Fresnel integrals, as is well known. Using this fact, it is possible to express v concisely by the following formula:

$$v(x,y) = e^{-iky} \left[F(r_2,\theta_2) - F(r_1,\theta_1) \right] - e^{iky} \left[F(r_2,\theta_2 + \pi) + F(r_1,\theta_1 + \pi) \right] ,$$

where

$$F(r,\theta) = \int_0^{2\sqrt{\frac{kr}{\pi}}\cos\theta/2} \exp\left|i\pi\tau^2/2\right| \, d\tau \quad ,$$

provided that the symbols r_1, r_2, θ_1, θ_2 are defined in Fig. 3, and that following conventions are observed in measuring θ_1, θ_2:

$$0 \leq \theta_2 < 2\pi, \quad 0 < \theta_1 \leq 2\pi \quad \text{for all } x \text{ when } \; y \leq 0 \; \text{ and for } |x| > a \text{ when } y \geq 0,$$

$$2\pi \leq \theta_1 \leq \frac{5\pi}{2}, \qquad \frac{3\pi}{2} \leq \theta_2 < 2\pi, \qquad\qquad 0 \leq x \leq a, \quad y \geq 0,$$

$$0 < \theta_1 \leq \frac{\pi}{2}, \qquad \frac{-\pi}{2} \leq \theta_2 \leq 0, \qquad\qquad -a \leq x \leq 0, \quad y \geq 0 \; .$$

The far-field behavior of the diffraction wave in angular directions bounded away from the vertical part of the T is obtainable in a simple form; we have

$$v \sim -\frac{(1+i)}{4\sqrt{\pi kR}} \; e^{ikR} \left\{ e^{-ika \sin\theta\left(\sec\frac{\theta}{2} + \operatorname{cosec}\frac{\theta}{2}\right)} - e^{ika \sin\theta\left(\sec\frac{\theta}{2} - \operatorname{cosec}\frac{\theta}{2}\right)} \right\}$$

In considering the diffraction of waves by obstacles it is usually assumed in the literature that the diffracting obstacles have zero thickness, in order that exact solution may be obtained. In practical problems, however, all obstacles have a finite thickness and a theory is needed to predict the effect of this discrepancy. It would be of interest to examine experimentally the diffraction of a plane wave incident normally on the short face of a long thick slab; perhaps it is possible to choose an equivalent width for the T obstacle considered here such that the present theoretical results show reasonable agreement with the experimental ones for a slab. If this is possible then the present results could be used to predict the effect of thickness for the thick slab. A similar comparison for oblique incidence might also be of interest. Here the field corresponding to a short strip would be involved, in addition to that for a half-plane.

4. Some remarks on eigenvalue problems

We now proceed to establish the equality of the eigenvalues for two different eigenvalue problems. We consider a region formed by a smooth curve C which is symmetric about two perpendicular diameters, and we define a function $u(x,y)$ to be an eigenfunction (for the Laplacian operator) which vanishes on C and also on one of the symmetrical radii, e.g., on $y = 0$, $x \geq 0$ (see Fig. 4), and is elsewhere a regular solution of $\Delta u + \lambda u = 0$.

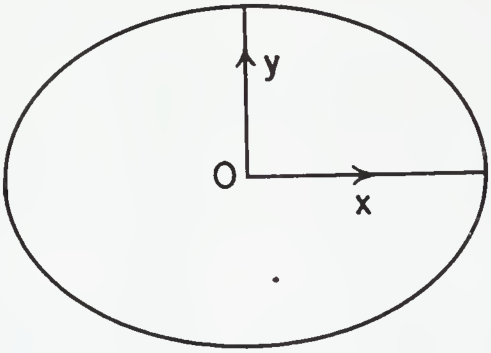


Figure 4

Proceeding as above we define a new function $w(x,y)$ as follows:

$$\begin{aligned} w(x,y) &= u(x,y) + u(-x,y), && y \leq 0, \\ &= u(x,y) - u(-x,y), && y \geq 0,\quad x \leq 0 \\ &= -u(x,y) + u(-x,y), && y \geq 0,\quad x \geq 0 \ . \end{aligned}$$

The function $w(x,y)$ satisfies the same equation as $u(x,y)$; however, it vanishes on a boundary composed of C and of $x = 0$, $y \geq 0$ and is elsewhere regular. Thus the eigenvalues for the original boundary are also eigenvalues for this second boundary. The original problem will be called problem A and the other problem B. It will now be shown that the first eigenvalue of problem A is also the first eigenvalue of problem B.

In order to do this it is necessary to examine in slightly greater detail the properties of nodal curves of eigenfunctions. The authors have been unable to find in the literature a proof that it is impossible for an eigenfunction to vanish only at one point in a two-dimensional domain. It is assumed outright that this is the case and that the nodal curves are smooth and continuous, of finite length, and do not terminate in the domain. Thus we assume that they must either form closed loops or terminate on the boundary of the domain.

It is sufficient to show that if u does not vanish neither does w, so that w is the lowest eigenfunction of problem B is u is the lowest eigenfunction of problem A. There are two ways in which w can vanish: either a) there exists a function $f_1(x) \leq 0$ such that $u(x, + f_1(x)) = - u(-x, f_1(x))$, or b) there exists a function $f_2(x) > 0$ such that $u(x, f_2(x)) = u(-x, f_2(x))$. The first alternative shows that u must change sign somewhere in the domain and hence it must vanish somewhere, in contradiction with the assumption that $u \neq 0$. To treat the possibility b) we consider the function $u(x,y) - u(-x,y)$; in view of b), $y = f_2(x)$ is a nodal curve for this function. That the latter is an eigenfunction of a proper subdomain of the original domain now follows from the assumed properties of nodal curves. All eigenvalues of such a subdomain are greater than the first eigenvalue for the larger original domain[4], and hence we again have a contradiction. Thus the first eigenvalues of problems A and B are equal.

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NYU
EM
83 Karp
AUTHOR

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TITLE
diffraction theory

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